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# Rephasing-Invariant Parametrizations of Generalized Kobayashi-Maskawa Matrices

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## Abstract

We discuss a simple rephase-invariant parametrization of the Kobayashi-Maskawa mixing matrix $\underline{V}$ which easily generalizes to more than three generations and which we believe to be suitable as a phenomenological standard. Our independent parameters are the magnitudes $|V_{i\alpha}|$ with $i < \alpha$ and the phases of plaquettes, $\arg( V_{i\alpha} V_{j\beta} V^*_{i\beta} V^*_{j\alpha} )$, where $j = i+1$, $\beta = \alpha+1$, and $j < \beta$. The detailed discussion includes consequences of unitarity constraints, modifications in cases of degenerate quark masses, and the relation to Jarlskog's invariant functions of mass matrices. We re-express the CP-violation phenomenology of the K- $\overline{K}$ and B- $\overline{B}$ systems in this rephase-invariant formalism. We exhibit a $4^{th}$ generation scenario where the top-quark mass need not be large even in the presence of large $B_d$- $\overline{B}_d$ mixing.

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## I. Introduction

In the standard model, CP violation is believed to be a consequence of complex values of elements of the 3x3 Kobayashi-Maskawa matrix[1] V which describe the couplings of the weak intermediate bosons $W^{\pm}$ to quarks. However, the phases of individual matrix elements of V are not themselves directly observable, because of arbitrariness in choice of phases of the quark fields. Therefore there is strong motivation to find a descriptive structure which is independent of such choices of phase. This problem has received a great deal of attention, and the 3x3 case is well-understood.[2] We are motivated to address this issue again mainly by curiosity on how the 3-generation description generalizes to n generations.[3] Here the situation is much less clear.

The description we offer does work in the n x n case, is reasonably simple and straightforward, and uses as raw material the quantities directly emergent from phenomenology. We believe it to be an especially suitable candidate for standardization of the phenomenology.

Our main suggestion is to replace the usual description of the Kobayashi-Maskawa matrix in terms of generalized Euler angles,[4] by a description using moduli of matrix elements and plaquette phases, defined below. The name "plaquette" is motivated by a rough analogy to gauge theories; the rephasing transformations play a role analogous to gauge transformations. As the definition suggests, the plaquette phases are then analogous to the field-strengths of gauge theories.

In the next section, we present the general description. In Section III we present details of the argument. In Section IV we discuss the cases of 3 and 4 generations. A graphical method used to describe unitarity constraints is discussed in Section V. Section VI touches on Jarlskog invariants, and parameterizations of mass degenerate cases are presented in Sect. VII. Rephasing-invariant phenomenology occupy Sect. VIII. Sect. IX concludes.

## II. The General Prescription

We label the n x n Kobayashi-Maskawa matrix $V_{i\alpha}$ with Latin indices for $Q=2/3$ quarks ($i=u,c,t,...$) and Greek indices for the $Q=-1/3$ quarks ($\alpha=d,s,b,...$). The number of independent real parameters characterizing the (unitary) V is $n^2$. Of these, $n(n-1)/2$ are "angle" parameters (this being the number of independent parameters for n x n real rotations). Of the 2n possible rephasings of the quark fields, one (a common phase change of all 2n quark fields) leaves V invariant. Hence the number of independent "phase" variables is

$$n^2 - \frac{n(n-1)}{2} - (2n-1) = \frac{(n-1)(n-2)}{2} \qquad (1)$$

A typical observable (in particular anything obtainable from Feynman-diagram calculations) will be a polynomial in V's and V*'s, with the restriction that in each term of the polynomial there be equal numbers of V's and V*'s, and that in each term the set of indices { i } in the product of V's be identical to the set { i } in the V*'s (This must of course also be true for the Greek indices { $\alpha$ }. ).

The simplest observable is the magnitude of each K-M element, $(V_{i\alpha}V^*_{i\alpha})^{1/2}$. The simplest which contains phase information is a product of four V's: $V_{i\alpha}\, V_{j\beta}\, V^*_{i\beta}\, V^*_{j\alpha}$. For the case $|i - j| = |\alpha-\beta| = 1$, we call this product a plaquette. The plaquettes, together with the $|V_{i\alpha}|^2$, will be our basic building blocks. We define

$$\Pi_{i\alpha} = V_{i\alpha}\, V_{i-1,\alpha-1}\, V^*_{i,\alpha-1}\, V^*_{i-1,\alpha} \qquad (2)$$

We furthermore define plaques as

$$^{j\beta}\Pi_{i\alpha} = V_{i\alpha}\, V_{j\beta}\, V^*_{i\beta}\, V^*_{j\alpha} \qquad (3)$$

As we demonstrate later, any observable consisting of a product of V's and

$V^*$'s can be written as a product of plaquettes, possibly multiplied by a product of $|V_{i\alpha}|$, and possibly divided by another product of $|V_{i\alpha}|$. (We assume, here and in what follows except Section. VII, that all elements of the K-M matrix are nonvanishing). It is therefore natural to associate the magnitudes of the $|V_{i\alpha}|$ with "angle" variables and the phases of the plaquettes (often just the imaginary part suffices) with the "phase" variables. In particular if we choose the $|V_{i\alpha}|$ with $\alpha > i$ as "angle" variables and also the arg $\Pi_{i\alpha}$ with $\alpha > i$ as the "phase" variables, the counting comes out correctly: there are n(n-1)/2 independent $|V_{i\alpha}|$ and (n-1) (n-2)/2 arg $\Pi_{i\alpha}$ (The topmost row with i = 1 is unavailable, and one has n(n -1) /2 - (n -1) = (n -1)(n -2) /2 elements remaining.).

This is our main proposition: <u>use the $|V_{i\alpha}|^2$ and arg $\Pi_{i\alpha}$ with $\alpha > i$ as the independent set of rephase-invariant variables</u>. We will show later that, given these parameters, the entire K-M matrix is determined up to the (2n-1) arbitrary quark-field phases, and up to a finite ambiguity which is no greater than $2^{n-2}$-fold, coming from solving quadratic equations in determining the magnitude of unknown diagonal V's. In the 3x3 case, this implies that $|V_{us}|^2$, $|V_{ub}|^2$, $|V_{cb}|^2$, and arg $\Pi_{cb}$ are the principal parameters. In the standard K-M parameterization, it is the imaginary part of the plaquette:[5]

$$J = \mathrm{Im}\,\Pi_{cb} = \mathrm{Im}\,V_{cb}\,V_{us}\,V^*_{cs}\,V^*_{ub} = s_1^2\,s_2\,s_3\,c_1\,c_2\,c_3\sin\delta \qquad (4)$$

which is the familiar and ubiquitous combination present in CP-violation phenomena. We note that

$$\mathrm{Im}\,\Pi_{cb} < |\,\Pi_{cb}| \le (.05)\mathrm{x}(0.2)\mathrm{x}1\mathrm{x}(.01) \sim 10^{-4} \qquad (5)$$

We also note the important result that in the 3x3 case,[5] all plaques have the same imaginary part. In fact, for all $i \neq j$; $\alpha \neq \beta$

$$\mathrm{Im}\, V_{i\alpha} V_{j\beta} V^*_{i\beta} V^*_{j\alpha} = \text{constant} = \pm J \tag{6}$$

This is a consequence of unitarity of the K-M matrix, and is discussed further in Section IV.

In the 4x4 case, the parameters are supplemented, in an obvious notation, by 5 new quantities, namely $|V_{uB}|^2$, $|V_{cB}|^2$, $|V_{tB}|^2$, arg $\Pi_{cB}$, and arg $\Pi_{tB}$. Thus, were new generations to emerge, the phenomenological structure need not undergo any major revision. New parameters become introduced and old unitarity constraints are modified. However the moduli and plaques associated below and on the diagonal, will be complicated functions[6] of the above parameters.[7]

III. Details

In order to substantiate the assertions of the previous section, it is necessary to first show that any (rephase-invariant) observable can be expressed in terms of $|V_{i\alpha}|$ and phases of plaquettes. Secondly, we have to show that given only the $|V_{i\alpha}|^2$ and arg $\Pi_{i\alpha}$ with $\alpha > i$ all remaining parameters of the K-M matrix are determined.

To demonstrate these assertions, it is useful to depict the observables, plaquettes, etc. which are products of V's and V*'s graphically.[8] The procedure is as follows:

i) If $V_{i\alpha}$ appears in the product, place an " o " in the $i\alpha$ entry of an originally empty n x n matrix. If $V^*_{i\alpha}$ appears, place an " x ".

ii) Then, from re-phase invariance, each row (or column) must have equal numbers of " x " and " o ".

iii) An " $\otimes$ " in a single given $i\alpha$ location is a factor $|V_{i\alpha}|^2$. These can be inserted or removed at will without changing the phase of the expression.

Now, given an arbitrary product $V_{i\alpha} \ldots V^*_{i\beta}$, which corresponds to a matrix with " x " and " o " entries, we may systematically eliminate the x's and the o's from the 1st column in terms of plaquettes, and then continue the procedure column by column. For example[9]

$$\arg \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ x & \bullet & o & \bullet \\ \bullet & \bullet & x & o \\ o & \bullet & \bullet & x \end{bmatrix} = \arg \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ x & \otimes & o & \bullet \\ \otimes & \otimes & x & o \\ o & \bullet & \bullet & x \end{bmatrix} = -\arg \Pi_{32} + \arg \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & x & o & \bullet \\ x & o & x & o \\ o & \bullet & \bullet & x \end{bmatrix} =$$

$$= -\arg \Pi_{32} - \arg \Pi_{33} + \arg \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ x & \bullet & \bullet & o \\ o & \bullet & \bullet & x \end{bmatrix} =$$

$$= \ldots \text{etc} \ldots = -\arg \Pi_{32} - \arg \Pi_{33} - \arg \Pi_{42} - \arg \Pi_{43} - \arg \Pi_{44} \qquad (7)$$

We trust the procedure is clear enough not to require the formal proof here. Therefore we argue it is possible to express all observables in terms of the magnitudes of K-M matrix-elements and the phases of plaquettes. What remains to be shown is that the limited set $\{ |V_{i\alpha}|, \arg \Pi_{i\alpha} \}$ with $\alpha > i$ suffices as well. We do this in several steps, by construction:

1) First choose the phases of $V_{1\alpha}$ and $V_{in}$. Since there are 2n-1 such elements, this exhausts the arbitrariness associated with rephasing of quark fields. (We shall return later to a suggestion for how this phase choice might most conveniently be made).

2) Use unitarity to determine $|V_{11}|$ and $|V_{nn}|$

$$|V_{11}|^2 = 1 - \sum_{\alpha>1} |V_{1\alpha}|^2 \tag{8a}$$

$$|V_{nn}|^2 = 1 - \sum_{i<n} |V_{in}|^2 \tag{8b}$$

3) At this point all elements in the top row and right-hand column are fully determined. Thus the phase of $V_{2,n-1}$ can be determined from the phase of the plaquette, $\Pi_{2n}$ in the upper-right hand corner of the matrix

$$\begin{bmatrix} * & * & * & * & * \\ \bullet & \bullet & \bullet & \bullet & * \\ \bullet & \bullet & \bullet & \bullet & * \\ \bullet & \bullet & \bullet & \bullet & * \\ \bullet & \bullet & \bullet & \bullet & * \end{bmatrix} \tag{9}$$

4) In the same way, the phases of the remaining $V_{2\alpha}$ in the second row with $\alpha \geq 2$ may be determined iteratively in terms of plaquette phases e.g.

$$\begin{bmatrix} * & * & * & * & * \\ \bullet & \bullet & * & * & * \\ \bullet & \bullet & \bullet & \bullet & * \\ \bullet & \bullet & \bullet & \bullet & * \\ \bullet & \bullet & \bullet & \bullet & * \end{bmatrix} \qquad (10)$$

Note that the phase of $V_{22}$ is determined at this stage, but not its magnitude.

5) The same procedure may be followed to determine the phases of all $V_{i\alpha}$ with $\alpha \geq i$:

$$\begin{bmatrix} * & * & * & * & * \\ \bullet & * & * & * & * \\ \bullet & \bullet & * & * & * \\ \bullet & \bullet & \bullet & * & * \\ \bullet & \bullet & \bullet & \bullet & * \end{bmatrix} \qquad (11)$$

although it must again be remembered that $|V_{ii}|$ is not determined for $1 < i < n$.

6) We now use unitarity to obtain the missing parameters in the 2nd row. Orthogonality of the first and second rows gives a linear relation between the (complex) $V_{21}$, the real $|V_{22}|$ and previously determined quantities

$$V_{21} V^*_{11} + V_{22} V^*_{12} + \sum_{\alpha=3}^{n} V_{2\alpha} V^*_{1\alpha} = 0 \qquad (12)$$

Now we may introduce the unitarity-constraint of normalization of the second row:

$$|V_{21}|^2 + |V_{22}|^2 = 1 - \sum_{\alpha=3}^{n} |V_{2\alpha}|^2 \qquad (13)$$

This is a quadratic equation in the unknown $|V_{22}|$. If the off-diagonal elements of the KM matrix are small (as in the case here), one root is positive (the physically correct solution). The other root will be negative, near -1, and thus physically unacceptable.[10]

7) This procedure can be again iterated. In the third row there are two orthogonality equations which determine the (complex) $V_{31}$ and $V_{32}$ as linear functions of $|V_{33}|$ with coefficients determined in terms of known quantities( up to the remote possibility of a twofold ambiguity in determining $|V_{22}|$ ). Normalization of the third row leads to a quadratic equation for $|V_{33}|$ with a remotely possible twofold ambiguity in its solution.

8) When we reach the $n^{th}$ row, the same procedure again may be used to determine $|V_{nn}|$. However $|V_{nn}|$ was already determined in step 2 without ambiguity. Thus no additional ambiguity is introduced at this stage, and it can be expected that the overall degree of ambiguity will, if present at all, be reduced. A highly conservative statement is that there is at most a $2^{n-2}$-fold ambiguity in reconstructing the K-M matrix from the input data. However, as long as the off-diagonal elements are as small as those seen experimentally, there will in fact be no ambiguity at all.

This completes the general argument on reconstruction of all K-M parameters from the input parameters. In the next section we will explicitly show how the procedure works for the 3 and 4 generation cases.

## IV. Parameterization

### A. Three Generations

The magnitudes of the K-M elements which serve for us as inputs are:[11]

$$|V_{us}| = 0.220 \pm .002 \tag{14a}$$

$$|V_{ub}| \leq 0.011 \text{ (90\% CL)} \tag{14b}$$

$$|V_{cb}| = 0.048 \pm .010 \tag{14c}$$

We suggest that for reconstruction purposes the 5 independent phase choices be made as follows:

(1) $V_{ud}$, $V_{us}$, $V_{cb}$, and $V_{tb}$ are chosen real and positive,

(2) The phase of $V^*_{ub}$ is chosen equal to the phase of the (only) input plaquette $\Pi_{cb}$.

$$\arg V^*_{ub} = \arg \Pi_{cb}. \tag{15}$$

This implies that $V_{cs}$ is also real and positive. Then we may proceed to reconstruct the remaining V's.

B. Four Generations

In the case for four generations, we proceed in a similar way. Again it will be convenient to choose phases such that the phases of plaquettes of interest are directly related to phases of the K-M matrix elements in the upper right-hand corner; i.e. $V_{ub}$, $V_{uB}$, and $V_{cB}$. We shall choose those such that their neighbors are real and positive. Specifically, the proposed generalization of the preceding section is as follows:

(1) Choose $V_{ud}$, $V_{us}$, $V_{tB}$, and $V_{TB}$ real and positive.

(2) As before, choose the phase of $V^*_{ub}$ equal to the phase of the plaquette $\Pi_{cb}$:

$$\arg V_{ub} = -\arg \Pi_{cb} \tag{16}$$

(3) In the same way choose the phase of $V^*_{cB}$ equal to the phase of the plaquette $\Pi_{tB}$:

$$\arg V_{cB} = -\arg \Pi_{tB} \tag{17}$$

(4) Finally choose the phase of $V_{uB}$ so that $V_{cb}$ remains real and positive. This is accomplished by the choice:

$$\arg V_{uB} = \arg V_{ub} + \arg V_{cB} - \arg \Pi_{cB}$$

$$= -\arg \Pi_{cb} - \arg \Pi_{tB} - \arg \Pi_{cB} \tag{18}$$

(5) From these definitions, it follows that, as in the 3x3 case, $V_{cs}$, $V_{cb}$, and $V_{tb}$ remain real and positive. The situation is shown schematically as

$$V = \begin{bmatrix} R & R & * & * \\ \bullet & (R) & (R) & * \\ \bullet & \bullet & (R) & R \\ \bullet & \bullet & \bullet & R \end{bmatrix} \tag{19}$$

where R denotes real-and-positive by definition, * denotes complex, and (R) denotes real-and-positive as a consequence of the phase choices made for the starred elements.

It is clear there is a useful generalization here: for the 2n-1 phase choices, take the $V_{ii}$ and $V_{i+1,i}$ to be real and positive. This puts the number of remaining phases for elements above the diagonal equal to the number of independent plaquette phases. The analysis of Section III remains valid with this convention. Indeed, we believe that if one insists on a standard phase convention, this one might be useful for phenomenology, since its connection to the rephase-invariant plaquettes is manifest.

## V. Modifications to the Relation $J = \pm \operatorname{Im} \Pi$ in Four Generations

We show some simple diagrammatics based on unitarity to prove the well known result $\operatorname{Im} \Pi = \pm J$ (eq.6) in the three generation case;[5] and then extend the diagrammatics to any number of generations. Define $\operatorname{Im} \Pi_{cs} = J$, then unitarity of KM matrix gives :

$$V_{11} V_{12}^* + V_{12} V_{22}^* + V_{13} V_{23}^* = 0 \tag{20}$$

One can write equivalently

$$\begin{bmatrix} o & \bullet & \bullet \\ x & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} + \begin{bmatrix} \bullet & o & \bullet \\ \bullet & x & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} + \begin{bmatrix} \bullet & \bullet & o \\ \bullet & \bullet & x \\ \bullet & \bullet & \bullet \end{bmatrix} = 0 \tag{21}$$

Multiplying by $V_{12}^* V_{22}$ we obtain

$$\begin{bmatrix} o & x & \bullet \\ x & o & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} + \begin{bmatrix} \bullet & \otimes & \bullet \\ \bullet & \otimes & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} + \begin{bmatrix} \bullet & x & o \\ \bullet & o & x \\ \bullet & \bullet & \bullet \end{bmatrix} = 0 \quad (22)$$

Taking the imaginary part removes the second term and we get

$$\mathrm{Im}\,\Pi_{22} = \mathrm{Im}\,\Pi_{23} = J. \quad (23)$$

In this fashion one easily sees that in three generations only one CP-sensitive parameter exists; $\mathrm{Im}\,\Pi = \pm J$.

The generalization to 4x4 matrices or higher creates a very large number of such relations. It is interesting to see how far one can go with these. Already for four generations, the large number of linear unitarity constraints which one can write down contain many which are linearly dependent. After detailed examination, it turns out that the nine $\mathrm{Im}\,\Pi_{i\alpha}$ for plaquettes can be expressed linearly in terms of nine other quantities which are the imaginary parts of big-plaques. By a big-plaque we mean a quantity

$$\text{big-plaque} = V_{i\alpha} V_{j\beta} V^*_{i\beta} V^*_{j\alpha} \quad (24)$$

with $|i - j| \geq 2$ and $|\alpha - \beta| \geq 2$

There are four 2x2 big-plaques, four 3x2 big-plaques and one 3x3 big-plaque. This re-expression of plaquettes can be useful because, under the assumption that $|V|$ decreases the farther it is from the diagonal, one relates phases of plaquettes on the diagonal to phases of elements, the moduli of which are small. Note that there is only one big-plaque which can reside in the 3 generation submatrix.

By repeated use of the unitarity condition,[12]

$$\mathrm{Im}\,\Pi_{22} = \mathrm{Im}\left\{ \begin{bmatrix} o & \bullet & x & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ x & \bullet & o & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} + \begin{bmatrix} o & \bullet & \bullet & x \\ \bullet & \bullet & \bullet & \bullet \\ x & \bullet & \bullet & o \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} + \begin{bmatrix} o & \bullet & x & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ x & \bullet & o & \bullet \end{bmatrix} + \begin{bmatrix} o & \bullet & \bullet & x \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ x & \bullet & \bullet & o \end{bmatrix} \right\}$$

$$\mathrm{Im}\,\Pi_{23} = \mathrm{Im}\,\Pi_{22} + \mathrm{Im}\left\{ \begin{bmatrix} \bullet & o & \bullet & x \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & x & \bullet & o \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} + \begin{bmatrix} \bullet & o & \bullet & x \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & x & \bullet & o \end{bmatrix} \right\}$$

$$\mathrm{Im}\,\Pi_{32} = \mathrm{Im}\,\Pi_{22} + \mathrm{Im}\left\{ \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ o & \bullet & x & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ x & \bullet & o & \bullet \end{bmatrix} + \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ o & \bullet & \bullet & x \\ \bullet & \bullet & \bullet & \bullet \\ x & \bullet & \bullet & o \end{bmatrix} \right\}$$

$$\mathrm{Im}\,\Pi_{33} = \mathrm{Im}\,\Pi_{23} + \mathrm{Im}\,\Pi_{32} - \mathrm{Im}\,\Pi_{22} + \mathrm{Im}\left\{ \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & o & \bullet & x \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & x & \bullet & o \end{bmatrix} \right\} \qquad (25)$$

In the limit of a trivial fourth-generation contribution (i.e no off-diagonal elements of $V_{j4}$ or $V_{4\alpha}$, this reduces immediately to the 3-generation case. To make good use of these relations, however, appears to require some knowledge of the fourth-generation K-M matrix elements. One can say that sufficient conditions for the 3-generation relations to survive are that $|V_{uB}|$, $|V_{cB}|$, $|V_{Td}|$, and $|V_{Ts}|$ all be small compared to $10^{-2}$.

## VI. Jarlskog invariants

Jarlskog, in an interesting paper,[13] pointed out that all physical quantities must be independent of an arbitrary unitary transformation affecting simultaneously the up and down quark mass matrices, denoted respectively m and m'. One diagonalizes the "square" of the mass-matrices via:[13,14]

$$U\ (m\, m^+)\ U^+ = D^2 \qquad (26a)$$

$$U'\ (m'\, m'^+)\ U'^+ = D'^2 \qquad (26b)$$

where U and U' are unitary matrices. The KM matrix is defined as:

$$V = UU'^+ \qquad (27)$$

There are $n^2+1$ physical measurables, 2n- quark masses and $(n-1)^2$ physical parameters of the mixing matrix. Jarlskog pointed out that physics does not change under the transformation

$$m\, m^+ \rightarrow X\, m\, m^+\, X^+ \qquad (28a)$$

$$m'm'^+ \rightarrow X\, m'm'^+\, X^+ \qquad (28b)$$

where X is an arbitrary unitary matrix. Under such a transformation, eq.28, the mass-eigenvalues of the up and down quarks and even the mixing matrix

$$V \rightarrow V \qquad (29)$$

stay invariant. As is well known not all the mixing matrix elements are physical quantities. The transformation discussed in previous Sections which leaves physics invariant, but changes the phases of KM elements is:

$$U \rightarrow T\, U \qquad (30a)$$

$$U' \rightarrow B^+\, U' \qquad (30b)$$

where T, B are arbitrary diagonal unitary matrices

$$V \rightarrow T\, V\, B. \qquad (31)$$

It appears that Jarlskog's approach includes all the physics, since one can not only express any |V|

$$| V_{i\alpha} |^2 = \mathrm{tr}\,\{ v_i(S)\, v'_\alpha(S') \} / ( \det v \ \det v' ) \qquad (32a)$$

but also any plaque as an invariant function of mass matrices, eq.28. In particular

$${}^{k\beta}\Pi_{i\alpha} = V_{i\alpha} V_{k\beta} V_{i\beta}^{*} V_{k\alpha}^{*} = V_{i\alpha} V_{k\alpha}^{*} V_{k\beta} V_{i\beta}^{*} = V_{i\alpha} V^{+}{}_{\alpha k} V_{k\beta} V^{+}{}_{\beta i} =$$

$$= \mathrm{tr}\,\{ E_i V E_\alpha V^+ E_k V E_\beta V^+ \} = \mathrm{tr}\,\{ v_i(S)\, v'_\alpha(S')\, v_k(S)\, v'_\beta(S') \} / ( \det v \ \det v' )^2 \qquad (32b)$$

here $S = m\, m^+$, $S' = m'\, m'^+$; $E_i$ and $E_\alpha$ are the elementary matrices; $v_i(S)$ and $v'_\alpha(S')$ are the Vandermonde-type matrices for the up and down sectors respectively (consult ref.13 for details[15]).

It appears that invariance under eq.28 can be likened to redefinition of fields and not to an underlying internal symmetry. A lucid example is the case of multiple scalar fields $\phi^t = ( \phi^1, ..., \phi^n )$ with a $\phi^4$ interaction. The lagrangian density reads:

$$L = 1/2 \{ \partial^\mu \phi^+ \partial_\mu \phi - \phi^+ m^2 \phi + \lambda( \phi^+ \phi)^2 \} \qquad (33)$$

The mass matrix can be diagonalized by a unitary transformation U, i.e.

$$U^+ m^2 U = D^2 \qquad (34)$$

Then under the redefinition of fields

$$\phi \to U \phi \qquad (35)$$

we will have a diagonalized lagrangian. Obtaining the "same physics" does not require identical actions $S = \int d^4x\, L$, but rather "same physics" falls into equivalence classes, definable by all lagrangians having identical $m^2$ eigenvalues ( not necessarily equal $m^2$ matrices ).

## VII. Degenerate masses

### A. Mass degeneracies

It is interesting to consider cases of d-fold degeneracies[16] in the up or down quark masses. The parameter counting goes as follows: One begins with $n(n-1)/2$ angle and $(n-1)(n-2)/2$ phase parameters. For each distinguishable d-fold degeneracy one subtracts by $d(d-1)/2$ both the angle and phase parameters. If a negative count of parameters results one takes 0 instead.

For instance, as is well known, in the three generation case existence of one two-fold degeneracy implies nonexistence of a CP-violating phase. For 4 generations and two 2-fold degeneracies, there remain 4 angle parameters and 1 phase, and not 5 angles and 0 phases.

Consider a d-fold mass degeneracy, and for concreteness take the first d up-quarks to be degenerate. Under a d X d unitary reshuffling U of the first d-rows of the KM matrix physics can not change.

$$V \quad \rightarrow \begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix} \quad V \tag{36}$$

Call the KM matrix restricted to the first d-rows $\overline{V}$. We observe that $\overline{V}^{+}\ \overline{V}$ is invariant under U-transformations eq.36 ( Trivial for $V^{+}V$.). The invariants under eq.36 are

$$\left( \overline{V}^{+}\ \overline{V} \right)_{\alpha\beta} = \sum_{i=1}^{d} \overline{V}^{*}_{i\alpha}\ \overline{V}_{i\beta} = \sum_{i=1}^{d} V^{*}_{i\alpha} V_{i\beta} \tag{37}$$

The summation extends only over the degenerate mass rows. Physically invariant quantities are obtained when we create rephasing invariant combinations in the down sector of $V^{+}V$, eq.37. For instance:

a) $\sum_{i=1}^{d} V^*_{i\alpha} V_{i\alpha}$ (38a)

b) $\sum_{i=1}^{d} V^*_{i\alpha} V_{i\beta} \,(V^*_{j\beta} V_{j\alpha}) = \sum_{i=1}^{d} {}^{i\beta}\Pi_{j\alpha}$ for $j > d$ (38b)

c) $| V_{j\alpha} |$ , ${}^{j\alpha}\Pi_{k\beta}$ for $j,k > d$ (38c)

It seems possible to relate any other physical quantity in terms of a)-c). For instance:

$$\sum_{i=1}^{d} V^*_{i\alpha} V_{i\beta} \sum_{j=1}^{d} V^*_{j\beta} V_{j\alpha} = \sum_{i,j=1}^{d} {}^{j\alpha}\Pi_{i\beta} = \left( \sum_{i,j=1}^{d} {}^{k\alpha}\Pi_{i\beta}\, {}^{j\alpha}\Pi_{r\beta} \right) / \, {}^{r\beta}\Pi_{k\alpha} \qquad \text{for } k, r > d \tag{39}$$

or:

$$\sum_{i=1}^{d} V^*_{i\alpha} V_{i\beta}\, V^*_{j\beta} V_{k\alpha}\, V^*_{k\gamma} V_{j\gamma} = {}^{k\alpha}\Pi_{j\gamma} \sum_{i=1}^{d} {}^{j\alpha}\Pi_{i\beta} \, / \, | V_{j\alpha} |^2 \qquad \text{for } k, j > d \tag{40}$$

As a physical parameterization we could choose the angle and phase parameters from the region bounded above by the $d^{th}$ row and bounded to the right by the diagonal; (the $d^{th}$ row and the diagonal are not included).

$$\begin{array}{c} d\left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\ \\ \\ \end{array} \begin{bmatrix} \backslash & \bullet & \bullet & \bullet & \bullet \\ \bullet & \backslash & \bullet & \bullet & \bullet \\ \bullet & \bullet & \backslash & \bullet & \bullet \\ \lozenge & \lozenge & \lozenge & \backslash & \bullet \\ \lozenge & \lozenge & \lozenge & \lozenge & \backslash \end{bmatrix} \tag{41}$$

The angle parameters, denoted by $\lozenge$, are taken as the magnitudes of KM elements in the above region. The phase parameters are taken as the arguments of all those plaquettes that involve at least three KM elements from the above region.

Indeed any plaque with two elements from the first d rows can always be rotated to 0, it has no physical significance. By a unitary transformation on the first d rows, eq.36, one can rotate all the first d rows below the diagonal simultaneously to 0. Therefore no angle or phase content is neglected in the parametrization, eq.41.

It is straightforward to analyze multiple different degeneracies. For example look at a 4-generation model where $m_u=m_c$ and $m_d=m_s$. Then just choose $|V_{33}|$, $|V_{34}|$, $|V_{43}|$, $|V_{44}|$ and arg $\Pi_{44}$ as our parameters. Indeed the submatrix

$$\begin{bmatrix} V_{33} & V_{34} \\ \\ V_{43} & V_{44} \end{bmatrix} \tag{42}$$

is not necessarily unitary and contains phase content.

B. Different number of up and down generations

Inspired by $E_6$ models[17] we consider the following. If we were to have an unequal number (n) of "up generations" and (m) of "down generations", then the mixing matrix could satisfy only one of the two equations:

a. $V\,V^{+}= 1_{n\,x\,n}$ (43a)

b. $V^{+}\,V = 1_{m\,x\,m}$ (43b)

Both unitarity conditions cannot be met simultaneously since the combined number of constraints $n^2 + m^2$ exceeds the initial number of real parameters 2nm characterizing an arbitrary complex n X m V-matrix. (In the square case (n=m): a $\Leftrightarrow$ b). To be definite take n < m; then the number of physical angle parameters is

$$n\,[\,(m-1)+(m-n)\,]\,/2 \tag{44a}$$

and phase parameters is

$$[\,(n-2\,)(m-1)+n(m-n)\,]\,/2 \tag{44b}$$

we assume a nondegenerate up-mass and down-mass spectrum. A physical parameterization can proceed as follows: take the region bounded to the left by the diagonal , which is not included.

$$n\left\{\begin{bmatrix} \backslash & \diamond & \diamond & \diamond \\ \bullet & \backslash & \diamond & \diamond \\ \bullet & \bullet & \backslash & \diamond \end{bmatrix}\right. \atop \underbrace{\hphantom{xxxxxxxxxxxxxx}}_{m} \qquad (45)$$

The angle parameters are the magnitudes of the KM elements in the bounded region, the phases are the arguments of the plaques constructed in that same region.
It is amusing that had we an m X m mixing matrix with n distinguishable up-quark masses [ a (m-n+1)-degeneracy]; then the surplus of angles and phases in the above mentioned n X m case differs by (m-n).

VIII. Rephase-Invariant Phenomenology

Here we display in rephasing invariant form[18] the kaon parameters $\Delta m$, $\Delta\Gamma$, $\varepsilon$, $\varepsilon'$ and look at the KM constraints from $K_L \to \mu^+\mu^-$ and from $B^0$- $\bar{B}^0$ and $D^0$- $\bar{D}^0$ mixings. In a later study we will include constraints coming from $K^+ \to \pi^+ \nu \bar{\nu}$ and from the electric dipole moment of the neutron.

A. Kaon system

We define the short and long lived species, assuming CPT-invariance, as:

$$| K_S > = p \,| K^0 > + q \,| \bar{K}^0 > \qquad (46a)$$

$$| K_L > = p \,| K^0 > - q \,| \bar{K}^0 > \qquad (46b)$$

The parameters p and q are not rephase-invariant. In the absence of CP violation the ratio p/q is of modulus unity. Define

$$e^{i\omega_0} \equiv \frac{<\pi\,\pi\,,\,I=0\,|\,H'\,|\,K^0>}{<\pi\,\pi\,,\,I=0\,|\,H'\,|\,\overline{K}^0>} \qquad (47)$$

The Wu-Yang phase convention, together with CP $| K^0 > = + | \overline{K}^0>$, implies $\omega_0$=0. Here we leave this phase, eq. 47, arbitrary. From their definitions it is clear that the following combinations are rephase-invariant

$$e^{i\omega_0} M_{12} = e^{i\omega_0} \sum_n \mathrm{P} \, \frac{<K^0\,|\,H'\,|\,n> \;\; <n\,|\,H'\,|\,\overline{K}^0>}{m_K - E_n} \qquad (48a)$$

$$e^{i\omega_0} \Gamma_{12} = e^{i\omega_0} \;\; 2\pi \sum_n \rho_n <K^0\,|\,H'\,|\,n> \;\; <n\,|\,H'\,|\,\overline{K}^0> \delta(m_K - E_n) \qquad (48b)$$

$$e^{-i\omega_0} \frac{q}{p} = e^{-i\,(\omega_0 + \arg \Gamma_{12})} \left( \frac{M^*_{12}/\Gamma^*_{12} - (i/2)}{M_{12}/\Gamma_{12} - (i/2)} \right)^{1/2} = e^{-i\,\omega_0} \frac{2\{\,M^*_{12} - (i/2)\,\Gamma^*_{12}\}}{-\Delta\lambda} \qquad (49)$$

where $\mathrm{P}$ stands for principal value and $\rho_n$ is the density of states. It is useful to recognize that $M_{12} / \Gamma_{12}$ is rephase-invariant and real in the limit of CP conservation. We now may express the (rephase-invariant) mass and lifetime difference of the kaons as[19]

$$\Delta m \approx -2 \;\mathrm{Re}\; (M_{12}\, e^{i\omega_0}) \qquad (50a)$$

$$\Delta\Gamma \approx -2 \;\mathrm{Re}\; (\Gamma_{12}\, e^{i\omega_0}) \qquad (50b)$$

Rephase-invariant definitions of the CP-violating parameters are

$$\varepsilon = \frac{<\pi\,\pi\,,\,I=0\,|\,H'\,|\,K_L>}{<\pi\,\pi\,,\,I=0\,|\,H'\,|\,K_S>} \qquad (51a)$$

and $$\varepsilon' = (1/\sqrt{2})\, i\, e^{\,i(\delta_2-\delta_0)}\, \mathrm{Im}\,(a_2/a_0) \qquad (51b)$$

Here we have defined

$$<\pi\,\pi\,,\,I\,|\,H'\,|\,K^0> = a_I\, e^{i\,\delta_I} \qquad (52)$$

It follows that:

$$\varepsilon \approx \frac{-\,i\ \mathrm{Im}\,(M_{12}\, e^{i\,\omega_0}) - \mathrm{Im}\,(\Gamma_{12}\, e^{i\,\omega_0})\,/\,2}{\Delta\lambda} \approx$$

$$\approx \frac{-\,i\ \mathrm{Im}\,(M_{12}\, e^{i\,\omega_0})}{\Delta\lambda} \qquad (51a)$$

Here $\Delta\lambda = \lambda_L - \lambda_S = \Delta m - i\,\Delta\Gamma/2$ are the eigenvalues of the 2 X 2 mass matrix $M\text{-}i\Gamma/2$. Eqs. 50-51 are all physical quantities and shown in a rephasing invariant way; under the rephasings

$$|\,K^0> \to e^{i\theta}\,|\,K^0> \qquad (53a)$$

$$|\,\overline{K^0}> \to e^{i\,\overline{\theta}}\,|\,\overline{K^0}> \qquad (53b)$$

physics does not change.

We remark that in the standard model the I=2 amplitude arises only[20] from the spectator diagram and hence[21]

$$a_2 = b\, V_{us}^{*}\, V_{ud}, \qquad (54a)$$

b being a real constant. On the other hand

$$a_0 = \sum_{q=u,c,t,\ldots} c_q V_{qs}^{*} V_{qd} \qquad (54b)$$

where $c_q$ are coefficients whose short distance contributions have been calculated.[22] We therefore obtain

$$\varepsilon' \approx \frac{1}{\sqrt{2}} \; i \; e^{i(\delta_2-\delta_0)} \; |a_2/a_0|^2 \; \frac{1}{|V_{us} V_{ud}|^2} (-) \, \mathrm{Im} \sum_{q=u,c,t,\ldots} (c_q/b) \; {}^{qd}\Pi_{us} \qquad (55)$$

Utilizing unitarity and assuming $(c_q/b)$ to be real we obtain in the four generation case

$$\varepsilon' \approx \frac{1}{\sqrt{2}} \; i \; e^{i(\delta_2-\delta_0)} \; |a_2/a_0|^2 \; \frac{1}{|V_{us} V_{ud}|^2 \; b} \{ -(c_T - c_c) \, \mathrm{Im}\, \Pi_{cs} + + (c_T - c_t) \; \mathrm{Im} \; {}^{us}\Pi_{td} \} \qquad (56)$$

One sees that $\varepsilon'/\varepsilon$ can be positive, negative, or even 0 and that $\varepsilon'/\varepsilon$ does not depend on the long-distance $c_u$ coefficient, which might harbor the $\Delta I=1/2$ rule explanation.

To calculate $\varepsilon$, one realizes that to a good approximation one can neglect the phase difference between $a_0$ and $a_2$. Then, exploiting the simple relation[20] ( eq.54a), we obtain[23]

$$M_{12} \, e^{i \, \omega_0} \sim \sum_{i,k=u,c,t,\ldots} {}^{ud}\Pi_{is} \; {}^{ud}\Pi_{ks} \; S\left(\frac{m_i^2}{M_W^2}, \frac{m_k^2}{M_W^2}\right) \qquad (57)$$

In the simple limit $x_i << x_k << 1$, $[\, x_i \equiv (m_i/ M_W)^2 \,]$ we have

$$S(x_i, x_k) \approx x_i \ln(x_k/x_i) \qquad (58a)$$

$$S(x_k) \equiv S(x_k, x_k) \approx x_k \qquad (58b)$$

For arbitrary quark masses exact expressions could be used.[24] $\varepsilon$ follows as[23]

$$\varepsilon \sim \frac{-i}{\Delta\lambda \; | V_{us} V_{ud} |^2} \sum_{i,k = u,c,t,\ldots} S( x_i, x_k ) \, \mathrm{Im} \{ {}^{ud}\Pi_{is} \; {}^{ud}\Pi_{ks} \} \qquad (59)$$

The $K_L$-$K_S$ mass difference is believed to arise mainly from long distance effects $K^0 \leftrightarrow 2\pi \leftrightarrow \overline{K}^0$. However, Gaillard and Lee[25] predicted the charm mass within the two generation GIM model.

$$\Delta m|_{exp} \sim m_c^2 \, \mathrm{Re} \, ({}^{ud}\Pi_{cs})^2 \approx m_c^2 \, \theta^4 \qquad (60)$$

As a rough upper bound, we note that higher generation contributions must not exceed $\Delta m$ and so must not exceed the charm contribution.

While the $\varepsilon'$ parameter involves the imaginary parts of the ${}^{qd}\Pi_{us}$ plaques(eq.55); the short distance contributions[26] of $K_L \rightarrow \mu^+ \mu^-$ contains information about their real parts. An estimate of the modulus of the short distance amplitude leads to:

$$| \mathrm{Re} \{ \Sigma_{q = u,c,t,\ldots} \; {}^{qd}\Pi_{us} \; m_q^2 \; \} | \leq | V_{us}^2 V_{ud} | \; 55 \, \mathrm{Gev}^2 \qquad (61)$$

Utilizing unitarity we may eliminate any one plaque; this yields for the four generation case:

$$| \; m_c^2 \, \mathrm{Re} \{ \; {}^{cd}\Pi_{us} \; \} + \; m_t^2 \, \mathrm{Re} \{ \; {}^{td}\Pi_{us} \; \} + m_T^2 \, \mathrm{Re} \{ \; {}^{Td}\Pi_{us} \; \} | \leq | V_{us}^2 V_{ud} | \; 55 \, \mathrm{Gev}^2 \qquad (62).$$

B. <u>B- $\overline{B}$ mixing</u>

The large $B_d$-mixing observed by the ARGUS collaboration[27]

$$(\Delta m/\gamma)_d \approx 0.7 \qquad (63)$$

implies, in the case of three generations,[28]

$$m_t \gtrsim 60 \text{ Gev.} \tag{64}$$

We review the reasoning as follows. To good approximation[28-31]

$$(\Delta m/\gamma)_d \approx .8\, B_d \left(\frac{f_{Bd}}{100 \text{ Mev}}\right)^2 \frac{S(x_t)}{S(m_t = 40 \text{ Gev})} \left|\frac{V_{td}}{V_{cb}}\right|^2 \tag{65}$$

where $B_d$ is the "bag-constant" and $f_{Bd}$ is the decay constant. From unitarity of the KM matrix and the experimental data one obtains that

$$| V_{td} / V_{cb} | \approx \theta \tag{66}$$

The easiest way to reconcile the experimental $B_d$-mixing result, eq.63, is to choose a larger top quark mass. This may not be obligatory given theoretical and experimental uncertainties. However in the four generation case, unitarity conditions are much relaxed. We know[11] from the B lifetime that

$$| V_{cb} | \approx \theta^2 \tag{67}$$

and from indirect unitarity bounds

$$| V_{td} | \leq 0.17 \ (95\% \text{ CL}), \quad 0 \leq | V_{tb} | < 1. \tag{68}$$

Assume, for the sake of an argument, that the top contribution is dominant even in the four generation scenario. Then one can easily fit the large $B_d$-mixing with small top quark masses (say 40 Gev ), by choosing $| V_{td} | \sim | V_{cb} | \sim \theta^2$.

Furthermore, were it to happen that $B_s$- $\overline{B}_s$ mixing is less than maximal, one would then have to look outside the standard 3 generation model.[30] One possible explanation could be found with 4 generations.[32]

C. Remarks & Speculations

Large 4th Generation mixing: An Example

To get a feel about the mixing magnitudes of a fourth generation consider

$$|V_{big}| \sim \begin{bmatrix} 1 & \theta & \theta^3 & \theta^2 \\ \theta & 1 & \theta^2 & \theta \\ \theta^3 & \theta^2 & 1 & \theta \\ \theta^2 & \theta & \theta & 1 \end{bmatrix} \tag{69}$$

We list a few consequences:

(a) $m_T \lesssim 30$ Gev ($K_L$-$K_S$ mass difference) ( 70a )

(b) $m_T \lesssim 34$ Gev ($K_L \rightarrow \mu^+\mu^-$) ( 70b )

(c) $m_B \lesssim 100$ Gev ($D^0$- $\bar{D}^0$ mixing) ( 70c )

(d) $m_T \sim 170$ Gev ( $B_d$- $\bar{B}_d$ mixing) ( 70d )

Remarks:

(a) In order that higher generation contributions not exceed the charm contribution to the $K_L$-$K_S$ mass difference, one must have

$$m_c^2\,\theta^4 \gtrsim m_T^2\,\theta^8 \tag{71}$$

leading to $m_T \lesssim 30$ Gev.

(b) The $K_L \rightarrow \mu^+\,\mu^-$ analysis (eq. 62 ) leads to eq. 70b.

(c) In the two generation case[33]

$$\frac{\Delta m_D}{\Delta m_K} = \left(\frac{f_D}{f_K}\right)^2 \quad \left(\frac{m_D}{m_K}\right) \quad \left(\frac{m_s}{m_c}\right)^2 \qquad (72)$$

With the experimental D-lifetime and $m_S$= 500 Mev one gets

$$(\Delta m/\gamma)_D = 10^{-3} \qquad (73)$$

Existence of an ultra heavy 4th generation B-quark leads to

$$\Delta m_D \sim [\,\theta^2 m_S{}^2 + m_B{}^2 \;(V_{cB} V^*_{uB})^2\,] \qquad (74)$$

$$(\Delta m/\gamma)_D \approx 10^{-3}\,[\,1+\theta^4\,(m_B/m_s)^2\,] \qquad (75)$$

From experiment[34] $(\Delta m/\gamma)_D \lesssim 10^{-1}$ hence eq. 70c.

(d) From $B_d$- $\overline{B}_d$ mixing, eq.63, and analogous reasoning to (c) we obtain[35] eq. 70d.

It appears that $V_{big}$, eq.69, is experimentally marginal.

## IX. Conclusions

The main purpose of this note is the proposed parametrization of the KM matrix. It is quite directly related to phenomenology, since the parameters consist of moduli of matrix elements and plaquette phases (defined in eq. 2 ); these are manifestly rephase-invariant. In the three-generation case, the question of how to parametrize the KM matrix is not too important. However if a generalization to a higher number of generations turns out to be necessary, the problem is less trivial.

If one does insist on a phase-dependent convention, we believe that choosing diagonal elements and those next-to diagonal elements which are above the diagonal to be real guarantees a simple relationship to plaquette phases. However one suffers from increasing complexity in the lower diagonal half. It may also be that experiment may dictate other choices; if a given set of $|V_{i\alpha}|$ are measured especially accurately, it makes sense to include them in the set of independent parameters. Likewise one might consider to use phases of those plaques directly related to the observed CP violation. Our basic point is to highlight the importance of rephase-invariance of any future parametrization, because only then is the physics manifest and not obscured by arbitrary phase conventions.

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$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ x & \bullet & o & \bullet \\ \bullet & \bullet & x & o \\ o & \bullet & \bullet & x \end{bmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ x & \bullet & o & \bullet \\ \otimes & \bullet & x & o \\ o & \bullet & \bullet & x \end{bmatrix} / |V_{td}|^2 = ({}^{21}\Pi_{33}\ {}^{31}\Pi_{44})^* \ / \ |V_{td}|^2$$

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